Analytic Pricing of Extinguishable Libor Cap-Floor Quanto

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Abstract

We seek analytic formulae for the pricing of extinguishable cap-floor Libor quanto flows which pay capped and/or floored foreign currency Libor conditional on survival of a named debt issuer. We assume that the foreign currency in which the Libor is paid and the credit default risk (observed for debt denominated in the domestic currency) are potentially correlated with each other and with the exchange rate between foreign and domestic currency; and further that this exchange rate may jump by a fixed relative amount in the event of a credit default event. The stochastic credit intensity model is assumed to be Black-Karasinski and the interest rate model to be beta-blend, a family of models which encompasses within its scope both the Black-Karasinski and the Hull-White cases.

The solutions are obtained as perturbation expansions which are valid in the limit of the foreign interest rate and the credit default intensity being small. A Green's function solution is found for the governing PDE which is asymptotically valid under this assumption. This is calculated to second order

and used to calculate to first order accuracy the price of extinguishable Libor cap-floor quanto flows, in particular taking account of the assumed correlations. When the interest rate model is Hull-White, a composite expansion is proposed which has the advantage of greater consistency with the known exact formula for the conditional bond price in the limit of zero correlation.

1 Introduction

We consider an economy with two currencies, foreign and domestic, and a credit default intensity process associated either with a corporation based in the foreign economy or a foreign sovereign issuer, the CDS market for which operates in the *domestic* currency. We suppose that the foreign interest rate is stochastic but, in the interests of tractability, that the domestic interest rate is deterministic. We also suppose that there is a stochastic exchange rate between the two currencies and further that upon default of the named issuer there is a *decrease* in the value of the foreign currency giving rise to a proportional downward jump in the exchange rate from foreign to domestic currency.

A similar model was first considered by Ehlers & Schönbucher (2006), who took the interest rate and credit intensity models to be affine, viz. in practice either of Hull & White (1990) or of Cox, Ingersoll & Ross (1985) (CIR) type. They did not solve directly the equations they derived but inferred jump levels from market data in the light of their modelling approach. Related work was reported by EL-Mohammadi (2009) who supposed the credit intensity to be lognormally distributed (with no mean reversion) and used his model to price defaultable FX options. A Black & Karasinski (1991) model was first considered in this context by Brigo et al. (2015) in their work reviewing the evidence from the comparative USD and EUR CDS rates for Italian sovereign debt of an implied FX jump in the wake of the euro crisis. Both the previous authors calculated prices using finite difference approaches. In more recent work by Itkin et al. (2017), both the interest rates were taken to be stochastic, governed by a CIR model, the foreign rate allowing of a jump at default as with the FX rate; the credit model was again taken to be Black-Karasinski.

A mathematical model effectively identical to that of Brigo et al. (2015) was proposed by Turfus (2016) to price contingent convertible (CoCo) bonds, with the equity price taking the role of the FX rate and a conversion intensity taking the role of the credit default intensity. Analytic formulae were derived in this case based on a perturbation expansion approach. Taking the lead from this work, we have like Itkin et al. (2017) sought to extend the modelling approach of Brigo et al. (2015) to include a stochastic (foreign) interest rate model, so as to allow cap-floor floating rate coupons to be priced analytically in a defaultable quanto framework. However the interest rate model is here assumed to be beta-blend, which approach encompasses both the Hull & White (1990) and Black & Karasinski (1991) short rate models. Thus we look to price in *domestic* currency an instrument which pays the foreign Libor rate, possibly capped and/or floored, conditional on survival of the named issuer. We will refer to an instrument which pays such cash flows as an extinguishable Libor cap-floor quanto (ELCFQ), and in the absence of any cap or floor as an ELQ.

We start in section 2 by defining the underlying processes in terms of suitably chosen auxiliary variables and the no-arbitrage conditions they are required to satisfy; we infer therefrom the governing PDE for contingent claims. Given the intractability of this PDE, we define in section 3 an asymptotic scaling of the underlying variables and associated functions based on the assumed smallness of the two short rates: foreign interest rate and credit default intensity. This gives rise to a tractable leading order PDE with the "intractable" parts isolated as a perturbation: we can then seek solutions as perturbation series. Rather than looking to obtain particular solutions directly, a Green's function for the full PDE is sought as a perturbation expansion, details of which are given in Appendix A. Consistency conditions are derived at this point to ensure the model satisfies the required no-arbitrage condition in relation to model-implied prices for risk-free and risky cash flows. This Green's function is then used in Appendix B to derive asymptotically valid prices for ELCFQ. The main results of the present work are set out in section 4, including some special case formulae applicable for the important Hull-White case. Some numerical tests of our formulae against results obtained by Monte Carlo simulation are presented in section 5. Conclusions are set out in section 6.

2 Stochastic Modelling

2.1 Definition of Underlying Processes

Our modelling approach is to represent the foreign interest rate and the credit default intensity as short rate diffusion processes and the FX rate as a jump diffusion. Specifically, we suppose the foreign interest rate process r_t to be governed for $t \geq 0$ by a beta-blend short rate model as set out in Turfus (2016), Horvath et al. (2017) and Turfus (2017). This encompasses both the Hull & White (1990) and Black & Karasinski (1991) short rate models. We suppose the credit default intensity process λ_t to be governed by a Black-Karasinski (lognormal) short rate model. The exchange rate Z_t from foreign into domestic currency we take to be given by a jump-diffusion process, with a downward jump of a fixed relative amount $k < 0$ occurring at the time of a default of the named issuer. Such jumps are commonly inferred from market data in relation to credit issuers which are perceived to carry systemic risk, particularly but not exclusively in the context of emerging markets. The diffusive processes are all potentially correlated. We effectively assume no correlation of any variables with the domestic interest rate, which justifies our taking it, for computational and notational convenience, to be deterministic.[1] Thus we take the domestic interest rate to be given by $r_d(t)$, assumed known from market data. We have in summary

$r_d(t)$: domestic interest rate;

r_t : foreign interest rate;

λ_t : credit default intensity for debt denominated in domestic currency;

Z_t : exchange rate from foreign to domestic currency.

We shall find it convenient to represent r_t and λ_t using auxiliary variables $\hat{x}_t$ and $\hat{y}_t$, respectively, satisfying the following Ornstein-Uhlenbeck processes:

$$d\hat{x}_t = -\alpha_r \hat{x}_t dt + \sigma_r(t)\, d\tilde{W}_t^1, \tag{2.1}$$

$$d\hat{y}_t = -\alpha_\lambda \hat{y}_t dt + \sigma_\lambda(t)\, dW_t^2, \tag{2.2}$$

where $d\tilde{W}_t^1$ and dW_t^2 are Brownian motions under the foreign currency and domestic currency equivalent martingale measures respectively with

$$\mathrm{corr}(\tilde{W}_t^1, W_t^2) = \rho_{r\lambda}$$

assumed and $\hat{x}_0 = \hat{y}_0 = 0$. These auxiliary variables are related to the foreign interest rate r_t and the credit default intensity λ_t, respectively, by

$$(1-\beta)\, r_t + \beta\overline{r}(t) = (\overline{r}(t) + (1-\beta) r^*(t))\mathcal{E}\left(\frac{(1-\beta)\hat{x}_t}{|\overline{r}(t)|^\beta}\right), \tag{2.3}$$

$$\lambda_t = (\overline{\lambda}(t) + \lambda^*(t))\mathcal{E}(\hat{y}_t), \tag{2.4}$$

with $\beta \in [0, 1)$ assumed. Here $\overline{r}(t)$ is the instantaneous forward rate of foreign interest and $\overline{\lambda}(t)$ the associated credit spread (see (2.15) below), with $\sigma_r(t)$ and $\sigma_\lambda(t)$ their respective volatilities and α_r and α_λ their mean reversion rates. Here $\mathcal{E}(X_t) := \exp\left(X_t - \frac{1}{2}[X]_t\right)$ is a stochastic exponential with $[X]_t$ the quadratic variation[2] of a process X_t under the requisite measure. The required form of the configurable functions $r^*(t)$ and $\lambda^*(t)$ is determined by calibration of the model to satisfy the no-arbitrage conditions set out below. As can be seen, the beta-blend model for the interest rate represents a hybrid between the Hull-White model ($\beta \uparrow 1$) and the Black-Karasinski model ($\beta = 0$). The credit intensity model is Black-Karasinski.

[1] In the absence of correlation between the domestic interest rate and any of the other three stochastic processes, stochasticity of the domestic interest rates can be shown to have no impact on the prices we calculate.

[2] Strictly, we will interpret $[X]_t$ in this definition as referring to the contribution only from the *diffusive* part of X_t, ignoring that arising from any jump component.

Finally we suppose the FX rate to be given by

$$\frac{dZ_t}{Z_t} = (r_d(t) - r_t - k\lambda_t)\,dt + \sigma_z(t)\,dW_t^3 + kdn_t. \tag{2.5}$$

where dW_t^3 is a Brownian motion and n_t a Cox process with intensity λ_t under the domestic currency equivalent martingale measure. We further suppose

$$\operatorname{corr}(\tilde{W}_t^1, W_t^3) = \rho_{rz},$$
$$\operatorname{corr}(W_t^2, W_t^3) = \rho_{\lambda z}.$$

It will be convenient to express Z_t also through an auxiliary variable, z_t, defined such that

$$Z_t = F(t)\mathcal{E}(z_t), \tag{2.6}$$

where

$$F(t) := Z_0 e^{\int_0^t (r_d(s) - \overline{r}(s) - k\overline{\lambda}(s))ds}, \tag{2.7}$$

whence we infer

$$dz_t = -\left(r_t - \overline{r}(t) + k(\lambda_t - \overline{\lambda}(t))\right)\,dt + \sigma_z(t)\,dW_t^3 + kdn_t. \tag{2.8}$$

with $z_0 = 0$.

We will also need the representation of $d\hat{x}_t$ under the *domestic* currency equivalent martingale measure. This is seen by application of the Girsanov theorem in the standard manner to be given by

$$d\hat{x}_t = -\alpha_r \hat{x}_t dt - \rho_{rz}\sigma_r(t)\sigma_z(t)dt + \sigma_r(t)\,dW_t^1, \tag{2.9}$$

for some new Brownian motion dW_t^1 related to $\tilde{W}_t^1$ by

$$d\tilde{W}_t^1 = dW_t^1 - \rho_{rz}\sigma_z(t)dt$$

This has solution subject to $\hat{x}_0 = 0$ given by

$$\hat{x}_t = -I_{rz}(t) + \int_0^t e^{-\alpha_r(t-s)}\sigma_r(s)dW_s^1 \tag{2.10}$$

where

$$I_{rz}(t) := \rho_{rz}\int_0^t e^{-\alpha_r(t-u)}\sigma_r(u)\sigma_z(u)\,du. \tag{2.11}$$

The no-arbitrage conditions

The formal no-arbitrage constraints which determine the functions $r^*(t)$ and $\lambda^*(t)$ are as follows. First, by considering a risk-free foreign currency cash flow at time t, we deduce

$$\mathbb{E}^f\left[e^{-\int_0^t r_s\,ds}\right] = D(0,t), \tag{2.12}$$

for $0 < t \leq T_m$, where T_m is the longest maturity date for which the model is calibrated, and

$$D(t_1, t_2) = e^{-\int_{t_1}^{t_2}\overline{r}(s)\,ds} \tag{2.13}$$

is the t_1-forward price of the t_2-maturity zero coupon bond denominated in foreign currency. Likewise, considering a *risky* domestic currency cash flow, we have

$$\mathbb{E}^d\left[e^{-\int_0^t (r_d(s)+\lambda_s)\,ds}\right] = B(0,t), \tag{2.14}$$

where

$$B(t_1, t_2) = e^{-\int_{t_1}^{t_2}(r_d(s)+\overline{\lambda}(s))\,ds} \tag{2.15}$$

is the price of a *risky* domestic currency cash flow. Here $\mathbb{E}^d[\cdot]$ and $\mathbb{E}^f[\cdot]$ represent expectations under the domestic and foreign currency martingale measures respectively. We shall assume the bond prices can be ascertained at the initial time $t = 0$ from the market, whence we can view (2.13) and (2.15) as implicitly *defining* $\overline{r}(t)$ and $\overline{\lambda}(t)$.

2.2 Derivation of Governing PDE

We consider in the first instance the general problem of pricing a cash security with maturity T whose payoff depends on $\hat{x}_T$, $\hat{y}_T$ and z_T. We introduce the convenient shorthand notation that, for a process X_t and deterministic function $f(.)$,

$$\mathcal{E}_x(f(t)X_t)) := \mathcal{E}(f(t)X_t))|_{X_t=x},$$

in terms of which we can re-write (2.3) and (2.4) as $r_t = r(\hat{x}_t, t)$ and $\lambda_t = \lambda(\hat{y}_t, t)$, where

$$r(\hat{x}, t) := \frac{1}{1-\beta}\left((\overline{r}(t) + (1-\beta)r^*(t))\mathcal{E}_{\hat{x}}\left(\frac{(1-\beta)\hat{x}_t}{|\overline{r}(t)|^\beta}\right) - \beta\overline{r}(t)\right), \tag{2.16}$$

$$\lambda(\hat{y}, t) := (\overline{\lambda}(t) + \lambda^*(t))\mathcal{E}_{\hat{y}}(\hat{y}_t), \tag{2.17}$$

Writing the price of the security at time $t \in [0, T]$ as $f_t^T = \hat{f}(\hat{x}_t, \hat{y}_t, z_t, t)$, we can infer by application of the Feynman-Kac theorem to (2.1), (2.2) and (2.8) in the standard manner that the function $\hat{f}(\hat{x}, \hat{y}, z, t)$ satisfies the following backward diffusion equation:

$$\left(\frac{\partial}{\partial t} + \hat{\mathcal{L}} - r_d(t) - \overline{\lambda}(t)\right)\hat{f}(\hat{x}, \hat{y}, z, t) = 0, \tag{2.18}$$

where

$$\begin{aligned}\hat{\mathcal{L}} :=& -\alpha_r\hat{x}\frac{\partial}{\partial\hat{x}} - \alpha_\lambda\hat{y}\frac{\partial}{\partial\hat{y}} - (r(\hat{x}, t) - \overline{r}(t) + k(\lambda(\hat{y}, t) - \overline{\lambda}(t)))\frac{\partial}{\partial z} - (\lambda(\hat{y}, t) - \overline{\lambda}(t))\cdot \\ &+ \frac{1}{2}\bigg(\sigma_r^2(t)\frac{\partial^2}{\partial\hat{x}^2} + \sigma_\lambda^2(t)\frac{\partial^2}{\partial\hat{y}^2} + \sigma_z^2(t)\frac{\partial^2}{\partial z^2} + 2\rho_{r\lambda}\sigma_r(t)\sigma_\lambda(t)\frac{\partial^2}{\partial\hat{x}\partial\hat{y}} + 2\rho_{rz}\sigma_r(t)\sigma_z(t)\frac{\partial}{\partial\hat{x}}\left(1 + \frac{\partial}{\partial z}\right) \\ &+ 2\rho_{\lambda z}\sigma_\lambda(t)\sigma_z(t)\frac{\partial^2}{\partial\hat{y}\partial z}\bigg)\end{aligned} \tag{2.19}$$

with in general $f_T^T = \hat{P}(\hat{x}_T, z_T)$ for some payoff function $\hat{P}(.)$.[3] In the absence of closed form solutions to (2.18) and guided by the work of Hagan et al. (2015) and Turfus (2017), we propose a perturbation expansion approach as follows.

3 Asymptotic Modelling

For both short rate models we formally apply a "low rates" assumption. To this end we define small parameters

$$\epsilon_r := \frac{1}{\alpha_r T_m}\int_0^{T_m}\overline{r}(t)\,dt, \tag{3.1}$$

$$\epsilon_\lambda := \frac{1}{\alpha_\lambda T_m}\int_0^{T_m}\overline{\lambda}(t)\,dt \tag{3.2}$$

and $O(1)$ functions

$$\begin{aligned}\tilde{r}(t) &:= \epsilon_r^{-1}\frac{\overline{r}(t)}{1-\beta}, \\ \tilde{r}^*(t) &; = \epsilon_r^{-1}r^*(t)\end{aligned} \tag{3.3}$$

[3] It is equally possible to handle a payoff on default within our proposed framework, although we shall not explicitly consider that possibility here.

and

$$\begin{aligned} \tilde{\lambda}(t) &= \epsilon_\lambda^{-1}\overline{\lambda}(t), \\ \tilde{\lambda}^*(t) &= \epsilon_\lambda^{-1}\lambda^*(t). \end{aligned} \tag{3.4}$$

Note, we make no formal requirement that either $\sigma_r(t)$ or $\sigma_\lambda(t)$ be small (scaled, say, in terms of their respective mean reversion rates). But, if they are, this will in general improve the accuracy of the perturbation expansions deduced. We further define a new scaled variable x_t and associated scaled volatility $\sigma_x(t)$ by

$$\begin{aligned} x_t &:= \epsilon_r^{-\beta}(\hat{x}_t + I_{rz}(t))e^{\alpha_r t}, \\ \sigma_x(t) &= \epsilon_r^{-\beta}\sigma_r(t)e^{\alpha_r t}. \end{aligned} \tag{3.5}$$

Notice that we can also write

$$\hat{x}_t := \epsilon_r^{\beta}e^{-\alpha_r t}(x_t + I_{xz}(0,t)), \tag{3.6}$$

with $I_{xz}(.)$ given by (A.12) below. Likewise we define

$$\begin{aligned} y_t &:= \hat{y}_t e^{\alpha_\lambda t}, \\ \sigma_y(t) &:= \sigma_\lambda(t)e^{\alpha_\lambda t}. \end{aligned} \tag{3.7}$$

Here the exponential time scaling is to facilitate removal of the mean reverting drift terms in (2.19). We further define new functional forms $f(\cdot)$ and $P(\cdot)$ by:

$$\begin{aligned} f(x,y,z,t) &:= f_t^T|_{x_t=x,y_t=y,z_t=z}, \\ P(x_T,z_T) &\equiv \hat{P}(\hat{x}_T,z_T), \end{aligned}$$

where $\hat{x}_t$ and $\hat{y}_t$ are related to x_t and y_t by (3.5) and (3.7), respectively for $0 < t \le \tau \wedge T$. In this notation, (2.1) and (2.2) can be re-expressed as

$$dx_t = \sigma_x(t)\,dW_t^1, \tag{3.8}$$

$$dy_t = \sigma_y(t)\,dW_t^2 \tag{3.9}$$

and (2.18) as

$$\left(\frac{\partial}{\partial t} + \mathcal{L} - (r_d(t) + \overline{\lambda}(t)) - (\epsilon_r h(x,t) + \epsilon_\lambda k g(y,t))\frac{\partial}{\partial z} - \epsilon_\lambda g(y,t)\right) f(x,y,z,t) = 0 \tag{3.10}$$

where

$$\mathcal{L}[\cdot\,] := \frac{1}{2}\Bigg(\sigma_x^2(t)\frac{\partial^2}{\partial x^2} + \sigma_y^2(t)\frac{\partial^2}{\partial y^2} + \sigma_z^2(t)\frac{\partial^2}{\partial z^2} + 2\rho_{r\lambda}\sigma_x(t)\sigma_y(t)\frac{\partial^2}{\partial x\partial y} + 2\rho_{rz}\sigma_x(t)\sigma_z(t)\frac{\partial^2}{\partial x\partial z} + 2\rho_{\lambda z}\sigma_y(t)\sigma_z(t)\frac{\partial^2}{\partial y\partial z}\Bigg) \tag{3.11}$$

is a pure diffusion and

$$h(x,t) := h(x,t,t), \tag{3.12}$$

$$g(y,t) := g(y,t,t), \tag{3.13}$$

with

$$h(x,t,t_1) := (\tilde{r}(t_1) + \tilde{r}^*(t_1))\mathcal{E}_x\left(F_\beta(t_1)(x_t - I_{xz}(0,t))\right) - \tilde{r}(t_1), \quad t_1 \ge t, \tag{3.14}$$

$$F_\beta(t) := \frac{(1-\beta)^{1-\beta}e^{-\alpha_r t}}{|\tilde{r}(t)|^\beta}, \tag{3.15}$$

$$g(y,t,t_1) := (\tilde{\lambda}(t_1) + \tilde{\lambda}^*(t_1))\mathcal{E}_y\left(e^{-\alpha_\lambda t_1}y_t\right) - \tilde{\lambda}(t_1), \quad t_1 \ge t. \tag{3.16}$$

We seek a Green's function solution for (3.10) as a joint power series in ϵ_r and ϵ_λ. Details are provided in Appendix A. This Green's function can then be used to generate asymptotic approximations to ELQ and ELCFQ products in Appendix B. The main conclusions of this analysis are summarised, in original unscaled notation, in section 4 below.

4 Main Results

4.1 Libor Flows

Theorem 4.1. *The price in domestic currency of an* Extinguishable Libor Quanto *payment at time T of tenor-τ foreign Libor fixed at time $T-\tau$ is under the modelling assumptions of section 2 given asymptotically in original unscaled variables for $0 \le \beta < 1$ by*

$$V_{ELQ} \sim \frac{F(T)B(0,T)}{D(T-\tau,T)} \left[(1-D(T-\tau,T))(1-(1+k)\phi_{\lambda z}(T)) - (1+k)\phi_{r\lambda z}(T-\tau,T)\right], \tag{4.1}$$

with errors $= \mathcal{O}(\epsilon_r(\epsilon_r + \epsilon_\lambda^2))$, where

$$\phi_{\lambda z}(T) := \int_0^T \overline{\lambda}(v)(e^{I_{\lambda z}(v)} - 1)\,dv, \tag{4.2}$$

$$\begin{aligned}\phi_{r\lambda z}(T_1,T_2) := &\int_{T_1}^{T_2} \overline{\lambda}(v)(e^{I_{\lambda z}(v)} - 1) \int_0^{T_2} \overline{r}(u) \frac{\exp(\hat{F}_\beta(u, u\wedge T_1)e^{-\alpha_\lambda(v-u\wedge T_1)} I_{r\lambda}(u\wedge T_1)) - 1}{1-\beta}\,du\,dv \\ &+ \int_0^{T_1} \overline{\lambda}(v) \int_{T_1}^{T_2} \overline{r}(u) \frac{\exp(\hat{F}_\beta(u,v) I_{r\lambda}(v)) - 1}{1-\beta}\,du\,dv\end{aligned} \tag{4.3}$$

$$\hat{F}_\beta(u,v) := \frac{(1-\beta)e^{-\alpha_r(u-v)}}{|\overline{r}(u)|^\beta}, \tag{4.4}$$

$$I_{r\lambda}(t) := \rho_{r\lambda} \int_0^t e^{-(\alpha_r+\alpha_\lambda)(t-u)} \sigma_r(u)\sigma_\lambda(u)\,du, \tag{4.5}$$

$$I_{\lambda z}(t) := \rho_{\lambda z} \int_0^t e^{-\alpha_\lambda(t-u)} \sigma_\lambda(u)\sigma_z(u)\,du. \tag{4.6}$$

Proof. The derivation of this result is covered in Appendices A and B, in particular at section B.1. □

Notice that, in the absence of any correlation involving the credit intensity, the $\phi.(\cdot)$ factors disappear and the standard result expected under the assumption of deterministic interest rates and credit default rates is recovered. The assumption of an exchange rate jump at default continues to give rise to the slight modification to the usual definition of the FX forward in (2.7). We infer from this that the $\mathcal{O}(\epsilon_r^n)$ errors propounded above disappear for the special case of $t = 0$, and the error in (4.1) is in fact $\mathcal{O}(\epsilon_r \epsilon_\lambda^2)$.

Although the expression (4.3) is not valid for the Hull-White case $\beta = 1$, it can yet be evaluated in the limit as $\beta \uparrow 1$. We have:

Corollary 4.1. *The result in Theorem 4.1 is also applicable for the case $\beta = 1$ provided* (4.3) *is replaced by its limiting value*

$$\begin{aligned}\phi_{r\lambda z}^{HW}(T_1,T_2) := &\int_{T_1}^{T_2} \overline{\lambda}(v)e^{-\alpha_\lambda(v-T_1)} \left(e^{I_{\lambda z}(v)} - 1\right) dv \int_0^{T_2} \gamma(u,T_1) I_{r\lambda}(u \wedge T_1)\,du \\ &+ B^*(T_2 - T_1) \int_0^{T_1} \overline{\lambda}(v) e^{-\alpha_r(T_1-v)} I_{r\lambda}(v) dv,\end{aligned} \tag{4.7}$$

with

$$B^*(\tau) := \frac{1-e^{-\alpha_r \tau}}{\alpha_r}. \tag{4.8}$$

$$\gamma(u,v) := \begin{cases} e^{-\alpha_\lambda(v-u)}, & u \le v, \\ e^{-\alpha_r(u-v)}, & u > v, \end{cases} \tag{4.9}$$

Proof. The above results are established by a straightforward application of l'Hôpital's theorem to the relevant integrands. $\square$

Note that a pure rates-credit correlation effect arises only from the second line of (4.3): a (small) non-zero impact arises from the first line only if credit intensity and FX are also correlated. For typical market parameters and investment grade credits, the PV impact from (4.7) is unlikely to exceed about 1% in relative terms, even with the most challenging case of long-dated 1y Libor rates.

Although we do not offer formal justification here, the structure of the Green's function generator in (A.1) suggests that the linear correction involving $\phi_{\lambda z}(\cdot)$ is in fact the leading order approximations to an exponential function. On this basis we propose that a potentially more accurate composite expansion can be obtained by completing the exponential as follows:

Proposition 4.1. *A better approximation than* (4.1) *should be achieved by using*

$$V_{ELQ} \sim \frac{F(T)B(0,T)}{D(T-\tau,T)} \left[e^{-(1+k)\phi_{\lambda z}(T)}(1-D(T-\tau,T)) - (1+k)\phi_{r\lambda z}(T-\tau,T) \right], \tag{4.10}$$

which has accuracy of the same order.

4.2 Impact of Caps and Floors

We consider next the impact of imposing a Libor cap on the value of an ELQ flow.

Theorem 4.2. *The price in domestic currency of a cap at level K_C applied to an* Extinguishable Libor Quanto *payment at time T of tenor-τ foreign Libor fixed at time $T-\tau$ is under the modelling assumptions of section 2 given asymptotically in original unscaled variables for $0 \le \beta < 1$ by*

$$\begin{aligned} V_{caplet} \sim \frac{F(T)B(0,T)}{D(T-\tau,T)} \Bigg[&(1-(1+k)\phi_{\lambda z}(T)) \Bigg((1-(1+K_C\delta(T-\tau,T))D(T-\tau,T))N(-\hat{d}_1(\hat{x}^*,T-\tau)) \\ &\qquad + \int_{T-\tau}^{T} \overline{r}(u) \frac{N(-\hat{d}_2(\hat{x}^*,u,T-\tau)) - N(-\hat{d}_1(\hat{x}^*,T-\tau))}{1-\beta} du \Bigg) \\ &- (1+k)\psi^{-}_{r\lambda z}(\hat{x}^*,T-\tau,T) \Bigg] \end{aligned} \tag{4.11}$$

with errors $= \mathcal{O}(\epsilon_r(\epsilon_r + \epsilon_\lambda^2))$*, where $\delta(\cdot)$ represents the day count fraction calculated according to the relevant*

convention,

$$\psi^{\pm}_{r\lambda z}(x,T_1,T_2) := \int_{T_1}^{T_2} \overline{\lambda}(v)\left(e^{I_{\lambda z}(v)}-1\right)\int_0^{T_2}\overline{r}(u) \frac{\exp(\hat{F}_\beta(u,u\wedge T_1)e^{-\alpha_\lambda(v-u\wedge T_1)}I_{r\lambda}(u\wedge T_1))N(\pm\hat{d}_3(\hat{x}^*,u,v,T_1)) - N(\pm\hat{d}_2(\hat{x}^*,u,T_1))}{1-\beta}\,du\,dv + \int_0^{T_1}\overline{\lambda}(v)\int_{T_1}^{T_2}\overline{r}(u)\frac{\exp(\hat{F}_\beta(u,v)I_{r\lambda}(v))N(\pm\hat{d}_3(\hat{x}^*,u,v,T_1)) - N(\pm\hat{d}_2(\hat{x}^*,u,T_1))}{1-\beta}\,du\,dv, \tag{4.12}$$

$$I_r(t) := \int_0^t e^{-2\alpha_r(t-u)}\sigma_r^2(u)\,du, \tag{4.13}$$

$$\hat{d}_1(x,T_1) := \frac{x}{\sqrt{I_r(T_1)}}, \tag{4.14}$$

$$\hat{d}_2(x,u,T_1) := \hat{d}_1(x) - \frac{\hat{F}_\beta(u+T_1,2(u\wedge T_1))I_r(u\wedge T_1)}{\sqrt{I_r(T_1)}}, \tag{4.15}$$

$$\hat{d}_3(x,u,v,T_1) := \hat{d}_2(x,u,T_1) - \frac{\hat{F}_\beta(u,u\wedge v\wedge T_1)e^{-\alpha_\lambda(v-u\wedge v\wedge T_1)}I_{r\lambda}(u\wedge v\wedge T_1)}{\sqrt{I_r(T_1)}}, \tag{4.16}$$

and $\hat{x}^*$ satisfies

$$\int_{T-\tau}^{T}\overline{r}(s)\exp(\hat{F}_\beta(s,T-\tau)\hat{x}^* - \tfrac{1}{2}\hat{F}_\beta^2(s)I_r(T-\tau))\,ds = (1-\beta)(1+K_C\delta(T-\tau,T))D(T-\tau,T)+\beta-D(T-\tau,T)^{-1}. \tag{4.17}$$

Proof. The derivation of this result is covered in Appendices A and B, in particular at section B.2. □

Note that the $\phi_{\lambda z}(\cdot)$ term can again if wished be replaced with an exponential. The Hull-White result is again obtained by letting $\beta \uparrow 1$:

Corollary 4.2. *The result corresponding to that in Theorem 4.2 for the case $\beta = 1$ is*

$$V^{HW}_{caplet} \sim \frac{F(T)B(0,T)}{D(T-\tau,T)}\Big[(1-(1+k)\phi_{\lambda z}(T))\Big((1-(1+K_C\delta(T-\tau,T))D(T-\tau,T))N(-\hat{d}_1(\hat{x}^*,T-\tau)) + B^*(\tau)\sqrt{I_r(T-\tau)}N'(-\hat{d}_1(\hat{x}^*,T-\tau))\Big) - (1+k)\psi^{-HW}_{r\lambda z}(\hat{x}^*,T-\tau,T)\Big] \tag{4.18}$$

where

$$\psi^{\pm HW}_{r\lambda z}(x,T_1,T_2) := \phi^{HW}_{r\lambda z}(T_1,T_2)\left(N(\pm\hat{d}_1(x,T_1)) \mp B^*(\tau)\sqrt{I_r(T-\tau)}N'(\pm\hat{d}_1(x,T_1))\right), \tag{4.19}$$

and the required value of $\hat{x}^*$ is

$$\hat{x}^* = \frac{(1+K_C\delta(T-\tau,T))D(T-\tau,T)-1}{B^*(\tau)}$$

Proof. This is a straightforward application of l'Hôpital's theorem. □

Alternatively, following the logic of Proposition 4.1 above and with an eye on the known exact Hull-White solution, we suggest:

Proposition 4.2. *A better approximation than* (4.18) *for the case* $\beta = 1$ *will be achieved by using the composite expansion*

$$V^{HW}_{caplet} \sim \frac{F(T)B(0,T)}{D(T-\tau,T)}\Big[e^{-(1+k)\phi_{\lambda z}(T)}\Big(N(-d_C^-) - (1+K_C\delta(T-\tau,T))D(T-\tau,T)N(-d_C^+)\Big) - (1+k)\phi^{HW}_{r\lambda z}(T-\tau,T)N(-d_C^-)\Big] \tag{4.20}$$

with

$$d_C^\pm := \frac{\ln(D(T-\tau,T)(1+K_C\delta(T-\tau,T))) \pm \frac{1}{2}B^*(\tau)^2 I_r(T-\tau)}{B^*(\tau)\sqrt{I_r(T-\tau)}}. \tag{4.21}$$

The expression (4.20) can be shown to agree with (4.18) to $\mathcal{O}(\epsilon_r\epsilon_\lambda)$, but has the desirable property of matching the known exact solution in the limit of zero correlation. Horvath et al. (2017) offer evidence that the difference between results arising from the two approaches is typically very small in practice. In fact, as with the result in Theorem 4.1, because the known exact result is recovered when the $\phi_\cdot(\cdot)$ factors go to zero, we infer that errors in (4.20) are likewise $\mathcal{O}(\epsilon_r\epsilon_\lambda^2)$.

It is also worth noting that, comparing (4.20) with the result arising in the absence of an assumed jump at default and stochastic credit, the impact of the jump at leading order is to multiply the effective credit intensity $\overline{\lambda}(t)$ by a factor of $1+k$, while the impact of FX-credit correlation is to multiply by a further factor of $e^{I_{\lambda z}(t)}$. This should be equally true for all values of β. The impact of rates-credit correlation is of course not so readily captured.

By a similar means or, more straightforwardly, by applying the put-call parity principle to (4.11), we see that the corresponding floorlet price for $\beta < 1$ with floor at K_F is, in original unscaled notation, given as follows:

Corollary 4.3. *The floor price corresponding to the situation in Theorem 4.2 is*

$$V_{floorlet} \sim \frac{F(T)B(0,T)}{D(T-\tau,T)}\Bigg[(1-(1+k)\phi_{\lambda z}(T))\Bigg(((1+K_F\delta(T-\tau,T))D(T-\tau,T)-1)N(\hat{d}_1(\hat{x}^*,T-\tau)) - \int_{T-\tau}^{T}\overline{r}(u)\frac{N(\hat{d}_2(\hat{x}^*,u,T-\tau)) - N(\hat{d}_1(\hat{x}^*,T-\tau))}{1-\beta}du\Bigg) + (1+k)\psi^+_{r\lambda z}(\hat{x}^*,T-\tau,T)\Bigg] \tag{4.22}$$

with errors $= \mathcal{O}(\epsilon_r(\epsilon_r+\epsilon_\lambda^2))$.

Note that the $\phi_{\lambda z}(\cdot)$ term can again if wished be replaced with an exponential. Similarly we have:

Proposition 4.3. *The floor price approximation corresponding to* (4.20) *for the case* $\beta = 1$ *is*

$$V^{HW}_{floorlet} = \frac{F(T)B(0,T)}{D(T-\tau,T)}\Big[e^{-(1+k)\phi_{\lambda z}(T)}\Big((1+K_F\delta(T-\tau,T))D(T-\tau,T)N(d_F^+) - N(d_F^-)\Big) + (1+k)\phi^{HW}_{r\lambda z}(T-\tau,T)N(d_F^-)\Big] \tag{4.23}$$

with

$$d_F^\pm := \frac{\ln(D(T-\tau,T)(1+K_F\delta(T-\tau,T))) = \pm\frac{1}{2}B^*(\tau)^2 I_r(T-\tau)}{B^*(\tau)\sqrt{I_r(T-\tau)}}. \tag{4.24}$$

Finally we observe that the value of an ELCFQ capped at K_C and floored at $K_F < K_C$ can be priced as a combination of an ELQ with a caplet and a floorlet, viz.

$$V_{\text{ELCFQ}} = V_{\text{ELQ}} - V_{\text{caplet}} + V_{\text{floorlet}}. \tag{4.25}$$

5 Numerical Tests

We test the formula (4.25) with (4.20) and (4.23) for an ECLFQ paying 3m Libor over five years. The foreign yield curve is assumed flat at 2.5%. A floor is imposed on Libor payments at 1.5% and various cap levels are considered. We take the Hull-White (normal) volatility to be $\sigma_r = 1\%$ with a mean reversion rate $\alpha_r = 0.25$, the 5y credit spread to be 40bp with a (lognormal) volatility of $\sigma_\lambda = 0.65$ and a mean reversion $\alpha_\lambda = 0.3$, and the FX volatility to be $\sigma_z = 0.25$. The correlation between FX and credit is taken to be $\rho_{\lambda z} = 0.4$ with a supposed downward jump of $k = 0.091$ in the FX rate at default. Comparisons between our analytic formulae and numerically computed prices based on Monte Carlo simulation are shown in Fig. 1. As can be seen the agreement is excellent and indeed comparable to the numerical error achieved with 500,000 MC simulations.

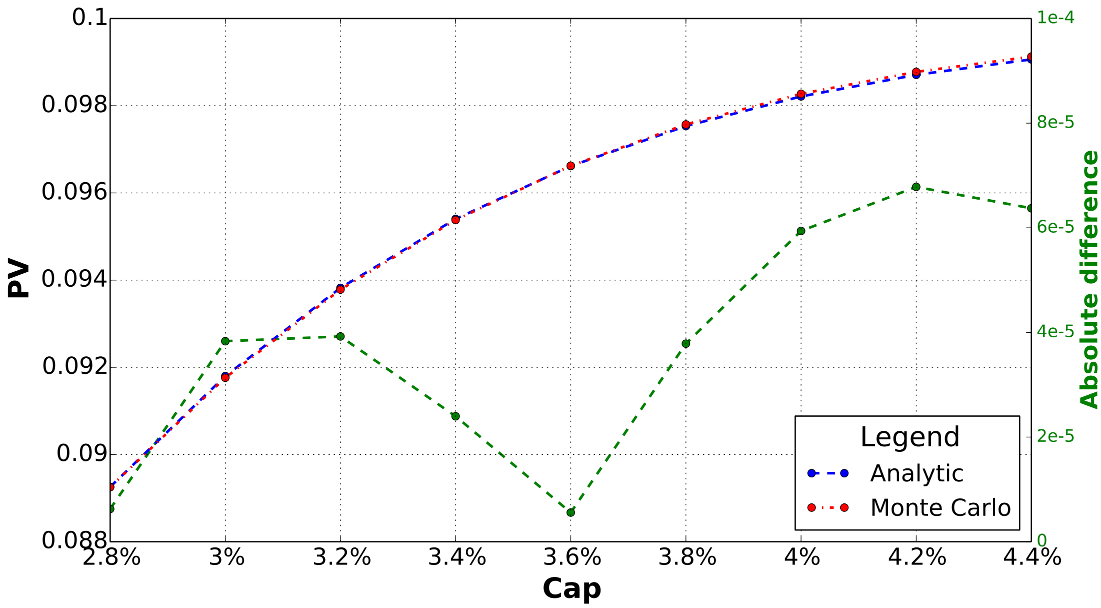

Figure 1: ELCFQ prices for various cap levels using Hull-White interest rate model.

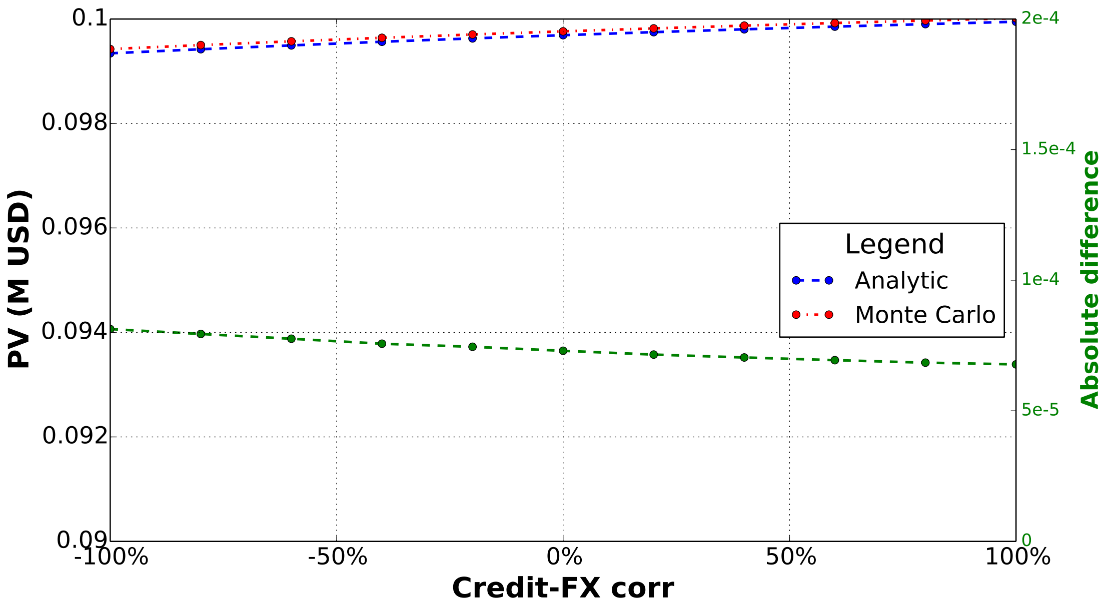

Figure 2: ELCFQ prices for various $\rho_{\lambda z}$ levels using Hull-White interest rate model.

A test is also done showing the impact of imposing a correlation $\rho_{\lambda z}$ between the credit intensity and the FX rate. The details of the calculation are as above except that no cap is imposed, only a floor. Results are illustrated in Fig. 2. The impact of the correlation appears to be well captured by our analytical model, to within the numerical error associated with the Monte Carlo simulation.

6 Conclusions

We have computed approximate formulae for the pricing in domestic currency of extinguishable cap-floor Libor quanto flows which pay capped and/or floored *foreign* currency Libor conditional on survival of a named debt issuer. We assumed that the foreign currency in which the Libor is paid and the credit default risk (observed for debt denominated in the domestic currency) are potentially correlated with the exchange rate between foreign and domestic currency. The domestic interest rate was taken for convenience to be deterministic. Solutions were obtained as perturbation expansions valid in the limit of the foreign interest rate and the credit default intensity being small. A Green's function solution to the governing PDE was derived accurate to second order in both parameters although in practice we only needed to use the first order terms of the Green's function to obtain what we believe will be adequate accuracy for many if not most circumstances.

In the absence of any cap or floor, the value of the Libor flow is given by (4.1) or (4.10). In the event that the flow is capped at a level K_C and/or floored at a level K_F, the value must be adjusted in accordance with (4.25). The caplet and floorlet formulae needed in this context are given by (4.11) and (4.22) above, respectively, for $\beta \in [0, 1)$. Variants of the formulae suitable for use in a Hull-White context ($\beta = 1$), which can be considered a limiting case of the aforementioned formulae, are given by (4.20) and (4.23). Comparisons of these formulae with results obtained by Monte Carlo simulation show them to be highly accurate.

A Green's Function

A.1 Leading Order Solution

We follow Hagan et al. (2015) in observing that the operator

$$\mathcal{U}(t;v) = \exp \int_t^v \left(\mathcal{L}(u) - \epsilon_r h(x,u) \frac{\partial}{\partial z} - \epsilon_\lambda g(y,u) \left(1 + k \frac{\partial}{\partial z}\right) \right) du \tag{A.1}$$

is a formal solution of

$$\left(\frac{\partial}{\partial t} + \mathcal{L} - \left(\epsilon_r h(x,t) + \epsilon_\lambda k g(y,t)\right) \frac{\partial}{\partial z} - \epsilon_\lambda g(y,t) \right) f(x,y,z,t) = 0 \tag{A.2}$$

for $t \leq v$ subject to $\mathcal{U}(v;v) = I$. The Green's function for this problem is then the integral kernel of $\mathcal{U}(t;v)$, viz.

$$G(x,y,z,t;\xi,\eta,\zeta,v) = \mathcal{U}(t;v)(x,y,z;\xi,\eta,\zeta). \tag{A.3}$$

The Green's function for (2.18) will then be given straightforwardly by

$$G^*(x,y,z,t;\xi,\eta,v) = B(t,v) G(x,y,z,t;\xi,\eta,\zeta,v). \tag{A.4}$$

We concentrate henceforth on the calculation of $G(\cdot)$. Again following Hagan et al. (2015) we consider first the limiting problem with $\epsilon_r = \epsilon_\lambda = 0$ as the leading order term for a perturbation expansion. The required solution can be written

$$\mathcal{U}_{0,0}(t;v) := \exp \int_t^v \mathcal{L}(u) du. \tag{A.5}$$

We will in all cases be interested in so-called free-boundary Green's function solutions which tend to zero as $x, y \to \pm\infty$. The Green's function solution subject to these conditions is well known. It is given by:

$$G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v) = \frac{\partial^3}{\partial\xi\partial\eta\partial\zeta} N_3(\xi - x, \eta - y, \zeta - z; R(t,v)), \quad t < v \tag{A.6}$$

where $N_3(x,y,z;R(t,v))$ is a trivariate Gaussian probability distribution function with mean $\mathbf{0}$ and covariance matrix

$$R(t,v) := \begin{pmatrix} I_x(t,v) & I_{xy}(t,v) & I_{xz}(t,v) \\ I_{xy}(t,v) & I_y(t,v) & I_{yz}(t,v) \\ I_{xz}(t,v) & I_{yz}(t,v) & I_z(t,v) \end{pmatrix} \tag{A.7}$$

with

$$I_x(t_1,t_2) := \int_{t_1}^{t_2} \sigma_x^2(u)\, du, \tag{A.8}$$

$$I_y(t_1,t_2) := \int_{t_1}^{t_2} \sigma_y^2(u)\, du, \tag{A.9}$$

$$I_z(t_1,t_2) := \int_{t_1}^{t_2} \sigma_z^2(u)\, du, \tag{A.10}$$

$$I_{xy}(t_1,t_2) := \rho_{r\lambda} \int_{t_1}^{t_2} \sigma_x(u)\sigma_y(u)\, du, \tag{A.11}$$

$$I_{xz}(t_1,t_2) := \rho_{rz} \int_{t_1}^{t_2} \sigma_x(u)\sigma_z(u)\, du, \tag{A.12}$$

$$I_{yz}(t_1,t_2) := \rho_{\lambda z} \int_{t_1}^{t_2} \sigma_y(u)\sigma_z(u)\, du. \tag{A.13}$$

Before deriving higher order terms in our proposed Green's function expansion, we introduce some additional notation. First, following Turfus (2017), we propose that the form of the configurable function $\tilde{r}^*(t)$ required to satisfy (2.12) is

$$\tilde{r}^*(t) = \sum_{i=1}^{\infty} \epsilon_r^i r_i^*(t). \tag{A.14}$$

Considering next the form of the expansion for $\tilde{\lambda}^*(t)$ required to satisfy (2.14), we note that Turfus (2017) considered the case where rates were stochastic. Here domestic rates are assumed deterministic so the calculation is on this occasion simpler (independent of the interest rate expansion). We are led to expect

$$\tilde{\lambda}^*(t) = \sum_{j=1}^{\infty} \epsilon_\lambda^j \lambda_j^*(t). \tag{A.15}$$

On this basis we can now expand $h(x, t, t_1)$ as

$$h(x, t, t_1) = \sum_{i=0}^{\infty} \epsilon_r^i h_i(x, t, t_1) - \tilde{r}(t_1) \tag{A.16}$$

where

$$\begin{aligned} h_0(x, t, t_1) &:= \tilde{r}(t_1)\mathcal{E}_x(F_\beta(t_1)(x_t - I_{xz}(0, t))) \\ h_i(x, t, t_1) &:= r_i^*(t_1)\mathcal{E}_x(F_\beta(t_1)(x_t - I_{xz}(0, t))), \quad i > 0. \end{aligned}$$

and similarly

$$g(y, t, t_1) = \sum_{j=0}^{\infty} \epsilon_\lambda^j g_j(x, t, t_1) - \tilde{\lambda}(t_1) \tag{A.17}$$

where

$$\begin{aligned} g_0(y, t, t_1) &:= \tilde{\lambda}(t_1)\mathcal{E}_y(e^{-\alpha_\lambda t_1} y_t) \\ g_j(y, t, t_1) &:= \lambda_j^*(t_1)\mathcal{E}_y(e^{-\alpha_\lambda t_1} y_t), \quad j > 0. \end{aligned}$$

Note that the required quadratic variations are given by $[x]_t = I_x(0, t)$ and $[y]_t = I_y(0, t)$. We further define

$$h_i(x, t) := h_i(x, t, t), \quad i \geq 0, \tag{A.18}$$

$$g_j(y, t) := g_j(y, t, t), \quad j \geq 0. \tag{A.19}$$

A.2 Asymptotic Expansion

Taking into consideration the structure of (3.10) and following Kato (1995, see in particular Chapter IX, §2), we propose that a formal asymptotic expansion of $\mathcal{U}(t; v)$ is possible as

$$\mathcal{U}(t; v) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \epsilon_r^i \epsilon_\lambda^j \mathcal{U}_{i,j}(t; v) \tag{A.20}$$

where each of the $\mathcal{U}_{i,j}(.)$ is $O(1)$. By a process of induction we infer that the following recurrence relation must hold:

$$\mathcal{U}_{i,j}(t; v) = -\int_t^v \mathcal{U}_{0,0}(t; u) \left(h(x, u) \frac{\partial}{\partial z} \mathcal{U}_{i-1,j}(u; v) \mathbb{1}_{i>0} + g(y, u) \left(1 + k \frac{\partial}{\partial z} \right) \mathcal{U}_{i,j-1}(u; v) \mathbb{1}_{j>0} \right) du, \quad i, j \geq 0. \tag{A.21}$$

Substituting this expression into (A.20) and and grouping like powers of ϵ_r and ϵ_λ, we can write

$$\mathcal{U}(t;v) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \epsilon_r^i \epsilon_\lambda^j \mathcal{U}^*_{i,j}(t;v),$$

where in particular we have

$$\begin{aligned}
\mathcal{U}^*_{0,0}(t;v) &= \mathcal{U}_{0,0}(t;v),\\
\mathcal{U}^*_{1,0}(t;v) &= -\int_t^v \mathcal{U}_{0,0}(t;u)(h_0(x,u)-\tilde{r}(u))\frac{\partial}{\partial z}\mathcal{U}_{0,0}(u;v)du,\\
\mathcal{U}^*_{0,1}(t;v) &= -\int_t^v \mathcal{U}_{0,0}(t;u)(g_0(y,u)-\tilde{\lambda}(u))\left(1+k\frac{\partial}{\partial z}\right)\mathcal{U}_{0,0}(u;v)du,\\
\mathcal{U}^*_{2,0}(t;v) &= -\int_t^v \mathcal{U}_{0,0}(t;u)\left((h_0(x,u)-\tilde{r}(u))\frac{\partial}{\partial z}\mathcal{U}_{1,0}(u;v)+h_1(x,u)\frac{\partial}{\partial z}\mathcal{U}_{0,0}(u;v)\right)du,\\
\mathcal{U}^*_{1,1}(t;v) &= -\int_t^v \mathcal{U}_{0,0}(t;u)\left((h_0(x,u)-\tilde{r}(u))\frac{\partial}{\partial z}\mathcal{U}_{0,1}(u;v)+(g_0(y,u)-\tilde{\lambda}(u))\left(1+k\frac{\partial}{\partial z}\right)\mathcal{U}_{1,0}(u;v)\right)du,\\
\mathcal{U}^*_{0,2}(t;v) &= -\int_t^v \mathcal{U}_{0,0}(t;u)\left((g_0(y,u)-\tilde{\lambda}(u))\left(1+k\frac{\partial}{\partial z}\right)\mathcal{U}_{0,1}(u;v)+g_1(y,u)\left(1+k\frac{\partial}{\partial z}\right)\mathcal{U}_{0,0}(u;v)\right)du.
\end{aligned}$$

Taking $G_{i,j}(.)$ to be the integral kernel of $\mathcal{U}^*_{i,j}$, we deduce:

$$G^*(x,y,z,t;\xi,\eta,\zeta,v) = B(t,v)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\epsilon_r^i\epsilon_\lambda^j G_{i,j}(x,y,z,t;\xi,\eta,\zeta,v). \tag{A.22}$$

We seek to obtain explicitly the form of the required integral kernels by a recursive process. We note in this regard that an analogous calculation was performed by Pagliarani et al. (2011) who addressed the problem of extending the Black-Scholes Green's function to address local volatility. At first order we obtain

$$\begin{aligned}
G_{1,0}(x,y,z,t;\xi,\eta,\zeta,v) &= -\int_t^v\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{0,0}(x,y,z,t;x_1,y_1,z_1,t_1)(h_0(x_1,t_1)-\tilde{r}(t_1))\\
&\qquad \frac{\partial}{\partial z_1}G_{0,0}(x_1,y_1,z_1,t_1;\xi,\eta,\zeta,v)\,dx_1dy_1dz_1dt_1\\
&= -\int_t^v (h_0(x,t,t_1)\mathcal{M}^x_{t,t_1}-\tilde{r}(t_1))\frac{\partial}{\partial z}G_{0,0}\left(x,y,z,t;\xi,\eta,\zeta,v\right)dt_1
\end{aligned} \tag{A.23}$$

and

$$\begin{aligned}
G_{0,1}(x,y,z,t;\xi,\eta,\zeta,v) &= -\int_t^v\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_{0,0}(x,y,z,t;x_1,y_1,z_1,t_1)(g_0(y_1,t_1)-\tilde{\lambda}(t_1))\\
&\qquad \left(1+k\frac{\partial}{\partial z_1}\right)G_{0,0}(x_1,y_1,z_1,t_1;\xi,\eta,\zeta,v)\,dx_1dy_1dz_1dt_1\\
&= -\int_t^v (g_0(y,t,t_1)\mathcal{M}^y_{t,t_1}-\tilde{\lambda}(t_1))\left(1+k\frac{\partial}{\partial z}\right)G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)\,dt_1,
\end{aligned} \tag{A.24}$$

where we have defined the shift operators

$$\mathcal{M}^x_{t_1,t_2}G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v) := G_{0,0}\left(x+F_\beta(t_2)I_x(t_1,t_2),y,z,t;\xi,\eta,\zeta,v\right) \tag{A.25}$$

$$\mathcal{M}^y_{t_1,t_2}G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v) := G_{0,0}\left(x,y+e^{-\alpha_\lambda t_2}I_y(t_1,t_2),z+I_{yz}(t_1,t_2),t;\xi,\eta,\zeta,v\right) \tag{A.26}$$

Similarly at second order we obtain after some manipulations

$$
\begin{aligned}
G_{2,0}(x,y,t;\xi,\eta,v) = & \int_t^v \int_t^{t_2} (h_0(x,t,t_1)\mathcal{M}^x_{t,t_1} - \tilde{r}(t_1))(h_0(x,t,t_2)\mathcal{M}^x_{t_1,t_2} - \tilde{r}(t_2)) \\
& \qquad \frac{\partial^2}{\partial z^2} G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)\, dt_1 dt_2 \\
& + \int_t^v (h_0(x,t,t_2) \int_t^{t_2} (h_0(x,t,t_1) \left(\exp\left(F_\beta(t_1)F_\beta(t_2)I_x(0,t_1)\right) - 1\right) \\
& \qquad \mathcal{M}^x_{t,t_1}\mathcal{M}^x_{t_1,t_2} \frac{\partial^2}{\partial z^2} G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)\, dt_1 dt_2 \\
& - \int_t^v h_1(x,t,t_1)\mathcal{M}^x_{t,t_1} \frac{\partial}{\partial z} G_{0,0}\left(x,y,z,t;\xi,\eta,\zeta,v\right) dt_1,
\end{aligned} \tag{A.27}
$$

$$
\begin{aligned}
G_{0,2}(x,y,t;\xi,\eta,v) = & \int_t^v \int_t^{t_2} (g_0(y,t,t_1)\mathcal{M}_{t,t_1} - \tilde{\lambda}(t_1))(g_0(y,t,t_2)\mathcal{M}_{t_1,t_2} - \tilde{\lambda}(t_2)) \\
& \qquad \left(1 + k\frac{\partial}{\partial z}\right)^2 G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)\, dt_1 dt_2 \\
& + \int_t^v (g_0(y,t,t_2) \int_t^{t_2} (g_0(y,t,t_1) \left(\exp\left(e^{-\alpha_\lambda(t_1+t_2)} I_y(0,t_1)\right) - 1\right) \\
& \qquad \left(1 + k\frac{\partial}{\partial z}\right)^2 \mathcal{M}_{t,t_1}\mathcal{M}_{t_1,t_2} G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)\, dt_1 dt_2 \\
& - \int_t^v g_1(y,t,t_1)\mathcal{M}_{t,t_1} \left(1 + k\frac{\partial}{\partial z}\right) G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)\, dt_1
\end{aligned} \tag{A.28}
$$

and

$$
\begin{aligned}
G_{1,1}(x,y,t;\xi,\eta,v) = & \int_t^v \int_t^{t_2} (h_0(x,t,t_1)\mathcal{M}^x_{t,t_1} - \tilde{r}(t_1))(g_0(y,t,t_2)\mathcal{M}^y_{t_1,t_2} - \tilde{\lambda}(t_2)) \\
& \qquad \left(1 + k\frac{\partial}{\partial z}\right) \frac{\partial}{\partial z} G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)\, dt_1 dt_2 \\
& + \int_t^v \int_t^{t_2} (g_0(y,t,t_1)\mathcal{M}^y_{t,t_1} - \tilde{\lambda}(t_1))(h_0(x,t,t_2)\mathcal{M}^x_{t_1,t_2} - \tilde{r}(t_2)) \\
& \qquad \left(1 + k\frac{\partial}{\partial z}\right) \frac{\partial}{\partial z} G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)\, dt_1 dt_2. \\
& + \int_t^v h_0(x,t,t_2) \int_t^{t_2} g_0(y,t,t_1) \left(\exp\left(e^{-\alpha_\lambda t_1} F_\beta(t_2) I_\rho(0,t_1)\right) - 1\right) \\
& \qquad \left(1 + k\frac{\partial}{\partial z}\right) \frac{\partial}{\partial z} \mathcal{M}^y_{t,t_1}\mathcal{M}^x_{t_1,t_2} G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)\, dt_1 dt_2 \\
& + \int_t^v g_0(x,t,t_2) \int_t^{t_2} h_0(x,t,t_1) \left(\exp\left(e^{-\alpha_\lambda t_2} F_\beta(t_1) I_\rho(0,t_1)\right) - 1\right) \\
& \qquad \left(1 + k\frac{\partial}{\partial z}\right) \frac{\partial}{\partial z} \mathcal{M}^x_{t,t_1}\mathcal{M}^y_{t_1,t_2} G_{0,0}(x,y,z,t;\xi,\eta,\zeta,v)\, dt_1 dt_2.
\end{aligned} \tag{A.29}
$$

Substituting (A.23)–(A.29) into (A.22) gives the required Green's function to second order accuracy.

As noted by Pagliarani et al. (2011), since the differential operators appearing in the Green's function apply only to z and not to ξ, η or ζ, they do not influence the applicability of the Green's function to the requisite payoff function, which consequently has no smoothness conditions imposed other than integrability,

say by its being piecewise continuous and exponentially bounded. As in the case of Pagliarani et al. (2011), it is if wished possible to apply the zero-order Green's function to the payoff and then apply the differential operator in z (and the integrations over time) thereafter. However, since the payoff functions we shall be considering here themselves rely on application of the Green's function to a pre-specified order of accuracy, this option is less relevant for us.

A.3 Calibration

It remains to calibrate our model consistent with the no-arbitrage conditions (2.12) and (2.14). This is achieved by considering in the former case the consistent pricing in domestic currency of a risk-free foreign currency cash flow, and in the latter case by consideration of a risky cash flow, as we shall now show.

We seek to calibrate the foreign interest rate model to ensure that (2.12) is satisfied. The necessary calculation for a risk-free cash flow in our model is equivalent to that performed by Turfus (2017) except here we express the results under the transformed variables for the domestic currency measure. We find the PV (in foreign currency) at time t is given by $f_t^T = X^T(x_t - I_{xz}(0,t), t)$ where

$$X^T(x,t) \sim D(t,T)\left(1 + \epsilon_r F_{1,0}(x,t) + \epsilon_r^2 F_{2,0}(x,t)\right) \tag{A.30}$$

with $O(\epsilon_r^3)$ errors, and

$$\begin{aligned}
F_{1,0}(x,t) &:= -\int_t^T \tilde{r}(t_1)(\mathcal{E}_x(F_\beta(t_1)x_t) - 1)\,dt_1 \\
F_{2,0}(x,t) &:= \tfrac{1}{2}F_{1,0}^2(x,t) + \int_t^T \mathcal{E}_x(F_\beta(t_2)x_t) \\
&\qquad \left(\tilde{r}(t_2)\int_t^{t_2} \tilde{r}(t_1)\mathcal{E}_x(F_\beta(t_1)x_t)\left(\exp\left(F_\beta(t_1)F_\beta(t_2)I_x(0,t_1)\right) - 1\right)dt_1 - r_1^*(t_2)\right)dt_2.
\end{aligned}$$

Of interest to us here is the conclusion that, for the expression in (A.30) to satisfy (2.12) above, we require that $F_{1,0}(0,0) = F_{2,0}(0,0) = 0$, so should choose

$$r_1^*(t) = \tilde{r}(t)\int_0^t \tilde{r}(u)\left(\exp\left(F_\beta(u)F_\beta(t)I_x(0,u)\right) - 1\right)du. \tag{A.31}$$

Higher terms will not be needed for the purposes of the present calculations wherein we truncate the Green's function at second order.

Pricing of risky domestic cash flow

The calibration of the credit intensity model proceeds by considering a risky domestic currency cash flow and, because domestic interest rates are assumed deterministic, is analogous to that set out above. For details, the reader can consult Turfus (2017). We state here only the conclusion, namely that satisfaction of (2.14) requires us to choose

$$\lambda_1^*(t) = \tilde{\lambda}(t)\int_0^t \tilde{\lambda}(u)\left(\exp\left(e^{-\alpha_\lambda(t+u)}I_y(0,u)\right) - 1\right)du, \tag{A.32}$$

Note the form of this is exactly analogous to (A.31) with $\beta \equiv 0$. This completes the calibration of our model to second order.

B Asymptotic Pricing

B.1 Extinguishable Libor Flow

We consider first the pricing of an ELCFQ *without* the cap/floor feature, denoting this $V_{\mathrm{ELQ}}(x_t, y_t, z_t, t)$. This can be modified to obtain the requisite ELCFQ price by subtracting the value of a caplet and adding

that of a floor, as we shall see below. We shall throughout make use of a first order Green's function approximation, i.e. using

$$G(\cdot) \sim G_{0,0}(\cdot) + \epsilon_r G_{1,0}(\cdot) + \epsilon_\lambda G_{0,1}(\cdot).$$

To the same order of approximation we can ignore the second order term $F_{2,0}(\cdot)$ in (A.30) above.

The payoff at time T for a payment of tenor-τ foreign currency Libor fixed at time $T-\tau$ can be written in domestic currency as

$$P_{\mathrm{ELQ}}(x_{T-\tau} = x, z_T = z) = F(T)\mathcal{E}_z(z_T)(X^T(x - I_{xz}(0, T-\tau), T-\tau)^{-1} - 1), \tag{B.1}$$

with $X^T(\cdot)$ defined by (A.30) above.[4]

Viewed from the perspective of time $T-\tau$, the payoff (B.1) is independent of both x and y, so its PV conditional on $(x_{T-\tau}, y_{T-\tau}, z_{T-\tau}) = (x, y, z)$ is seen by straightforward application of our first order Green's function to be

$$\begin{aligned}
V_{\mathrm{ELQ}}(x, y, z, T-\tau) = {} & F(T)\mathcal{E}_z(z_{T-\tau})\frac{B(T-\tau, T)}{D(T-\tau, T)} \\
& \left(1 - \epsilon_\lambda(1+k)\int_{T-\tau}^{T}\left(g_0(y, T-\tau, v)\exp\left(e^{-\alpha_\lambda v} I_{yz}(T-\tau, v)\right) - \tilde{\lambda}(v)\right) dv\right) \\
& \left(1 - D(T-\tau, T) + \epsilon_r \int_{T-\tau}^{T} (h_0(x, T-\tau, u) - \tilde{r}(u))\, du\right) \\
& + \mathcal{O}(\epsilon_r(\epsilon_r + \epsilon_\lambda^2)),
\end{aligned}$$

where the $\mathcal{O}(\epsilon_r^2)$ errors arise on account of us having chosen to ignore second order terms in the payoff from the outset. We can now treat this expression as the payoff at time $T-\tau$ for $V_{\mathrm{ELQ}}(x, y, z, t)$ for $t < T-\tau$. We are interested in particular in the price at $t = 0$. Applying our first order Green's function to this final value problem we obtain

$$\begin{aligned}
V_{\mathrm{ELQ}}(t=0) &\sim \sum_{i,j\geq 0}^{i+j\leq 1} \epsilon_r^i \epsilon_\lambda^j B(0, T-\tau) \iiint_{\mathbb{R}^3} G_{i,j}(0,0,0,0;\xi,\eta,\zeta,T-\tau)\, V_{\mathrm{ELQ}}(\xi,\eta,\zeta,T-\tau)\, d\xi\, d\eta\, d\zeta \\
&\sim \frac{F(T)B(0,T)}{D(T-\tau,T)}\Big((1 - D(T-\tau,T))(1 - \epsilon_\lambda(1+k)\phi_{yz}(T)) - \epsilon_r\epsilon_\lambda(1+k)\phi_{xyz}(T-\tau, T)\Big)
\end{aligned} \tag{B.2}$$

with errors $= \mathcal{O}(\epsilon_r(\epsilon_r + \epsilon_\lambda^2))$, where

$$\phi_{yz}(T) := \int_0^T \tilde{\lambda}(v))\left(\exp\left(e^{-\alpha_\lambda v} I_{yz}(0, v)\right) - 1\right) dv, \tag{B.3}$$

$$\begin{aligned}
\phi_{xyz}(T_1, T_2) := {} & \int_{T_1}^{T_2} \tilde{\lambda}(v)\left(\exp\left(e^{-\alpha_\lambda v} I_{yz}(0, v)\right) - 1\right) \int_{T_1}^{T_2} \tilde{r}(u)\left(\exp\left(F_\beta(u) e^{-\alpha_\lambda v} I_{xy}(0, T_1)\right) - 1\right) du\, dv \\
& + \int_{T_1}^{T_2} \tilde{\lambda}(v)\left(\exp\left(e^{-\alpha_\lambda v} I_{yz}(0, v)\right) - 1\right) \int_0^{T_1} \tilde{r}(u)\left(\exp\left(F_\beta(u) e^{-\alpha_\lambda v} I_{xy}(0, u)\right) - 1\right) du\, dv \\
& + \int_0^{T_1} \tilde{\lambda}(v) \int_{T_1}^{T_2} \tilde{r}(u)\left(\exp\left(F_\beta(u) e^{-\alpha_\lambda v} I_{xy}(0, v)\right) - 1\right) du\, dv.
\end{aligned} \tag{B.4}$$

[4]We ignore here any possible basis between the Libor curve and that for the risk-free rate captured by r_t. However, it is not difficult to deal with such a basis by using a discount curve which represents the foreign currency tenor-τ Libor rate rather than the risk-free rate, since for discounting purposes we always use the *domestic* currency curve (and we suppose there are no domestic currency Libor payments).

Here the first line in the expression for $\phi_{xyz}(\cdot)$ arises from $G_{0,0}$, the second from $G_{1,0}$ and the third from $G_{0,1}$. This can be rewritten more compactly as

$$
\begin{aligned}
\phi_{xyz}(T_1,T_2) := & \int_{T_1}^{T_2} \tilde{\lambda}(v)\left(\exp\left(e^{-\alpha_\lambda v} I_{yz}(0,v)\right)-1\right)\int_0^{T_2} \tilde{r}(u)\left(\exp\left(F_\beta(u)e^{-\alpha_\lambda v} I_{xy}(0,u\wedge T_1)\right)-1\right) du\,dv \\
& + \int_0^{T_1} \tilde{\lambda}(v) \int_{T_1}^{T_2} \tilde{r}(u)\left(\exp\left(F_\beta(u)e^{-\alpha_\lambda v} I_{xy}(0,v)\right)-1\right) du\,dv.
\end{aligned}
\tag{B.5}
$$

Expressions in original unscaled notation for the PV of Extinguishable Libor Flows are presented in section 4.1 above.

B.2 Extinguishable Caplet

To calculate the impact of the Libor calculated above being capped at K_C, we must compute the value of a caplet which pays the amount by which the Libor exceeds K_C, if this is positive, conditional on survival of the named issuer, multiplied by the relevant day count fraction. Thus we have a payoff at time T given by

$$
P_{\text{caplet}}(x_{T-\tau}=x, z_T=z) = F(T)\mathcal{E}_z(z_T)(X^T(x-I_{xz}(0,T-\tau),T-\tau)^{-1}-\kappa^{-1})\mathbb{1}_{X^T(x-I_{xz}(0,T-\tau),T-\tau)\le\kappa}, \tag{B.6}
$$

where

$$
\kappa := \frac{1}{1+K_C\,\delta(T-\tau,T)}.
$$

Proceeding as previously with first order representations, we deduce our first order solution as of time $T-\tau$ to be

$$
\begin{aligned}
V_{\text{caplet}}(x,y,z,T-\tau) = & F(T)\mathcal{E}_z(z_{T-\tau})\frac{B(T-\tau,T)}{D(T-\tau,T)}\,\mathbb{1}_{x>x^*+I_{xz}(0,T-\tau)} \\
& \left(1-\epsilon_\lambda(1+k)\int_{T-\tau}^{T}\left(g_0(y,T-\tau,v)\exp\left(e^{-\alpha_\lambda v}I_{yz}(T-\tau,v)\right)-\tilde{\lambda}(v)\right)dv\right) \\
& \left(1-\kappa^{-1}D(T-\tau,T)+\epsilon_r\int_{T-\tau}^{T}(h_0(x,T-\tau,u)-\tilde{r}(u))\,du\right) \\
& + \mathcal{O}(\epsilon_r(\epsilon_r+\epsilon_\lambda^2)),
\end{aligned}
$$

where x^* is the unique solution of

$$
\epsilon_r\int_{T-\tau}^{T}(h_0(x+I_{xz}(0,T-\tau),T-\tau,t_1)-\tilde{r}(t_1))\,dt_1 = \kappa^{-1}D(T-\tau,T)-1.
$$

Applying our first order Green's function with the $T-\tau$ solution as payoff, we obtain the following expression for the PV as of $t=0$:

$$
\begin{aligned}
V_{\text{caplet}}(t=0) \sim & \sum_{i,j\geq 0}^{i+j\leq 1} \epsilon_r^i \epsilon_\lambda^j B(0,T-\tau) \iiint_{\mathbb{R}^3} G_{i,j}(0,0,0,0;\xi,\eta,\zeta,T-\tau)\, V_{\text{caplet}}(\xi,\eta,\zeta,T-\tau)\, d\xi\, d\eta\, d\zeta \\
\sim & \frac{F(T)B(0,T)}{D(T-\tau,T)} \Bigg[(1-\epsilon_\lambda(1+k)\phi_{yz}(T)) \Bigg((1-\kappa^{-1} D(T-\tau,T)) N(-d_1(x^*,T-\tau)) \\
& + \epsilon_r \int_{T-\tau}^{T} \tilde{r}(u) \Big(N(-d_2(x^*,u,T-\tau)) - N(-d_1(x^*,T-\tau)) \Big) du \Bigg) \\
& - \epsilon_r \epsilon_\lambda (1+k) \psi_{xyz}(x^*,T-\tau,T) \Bigg]
\end{aligned} \tag{B.7}
$$

with $\mathcal{O}(\epsilon_r(\epsilon_r+\epsilon_\lambda^2))$ error, where

$$
\begin{aligned}
\psi_{xyz}(x^*,T_1,T_2) = & \int_{T_1}^{T_2} \tilde{\lambda}(v) \left(\exp\left(e^{-\alpha_\lambda v} I_{yz}(0,v) \right) - 1 \right) \int_0^{T_2} \tilde{r}(u) \\
& \left(\exp\left(F_\beta(u) e^{-\alpha_\lambda v} I_{xy}(0,u\wedge T_1) \right) N(-d_3(x^*,u,v,T_1)) - N(-d_2(x^*,u,T_1)) \right) du\, dv \\
& + \int_0^{T_1} \tilde{\lambda}(v) \int_{T_1}^{T_2} \tilde{r}(u) \\
& \left(\exp\left(F_\beta(u) e^{-\alpha_\lambda v} I_{xy}(0,v) \right) N(-d_3(x^*,u,v,T_1)) - N(-d_2(x^*,u,T_1)) \right) du\, dv
\end{aligned} \tag{B.8}
$$

and

$$
d_1(x,T_1) := \frac{x}{\sqrt{I_x(0,T_1)}}, \tag{B.9}
$$

$$
d_2(x,u,T_1) := d_1(x) - \frac{F_\beta(u) I_x(0,u\wedge T_1)}{\sqrt{I_x(0,T_1)}}, \tag{B.10}
$$

$$
d_3(x,u,v,T_1) := d_2(x,u,T_1) - \frac{F_\beta(u) e^{-\alpha_\lambda v} I_{xy}(0,u\wedge v\wedge T_1)}{\sqrt{I_x(0,T_1)}} \tag{B.11}
$$

Note that by construction

$$
\lim_{x^*\to -\infty} \psi_{xyz}(x^*,T_1,T_2) = \phi_{xyz}(T_1,T_2) \tag{B.12}
$$

Analogous results can be derived for floorlets by appeal to the put-call parity theorem. Expressions in original unscaled notation for both caplets and floorlets are presented in section 4.2 above.

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